On Holder-Lipschitz condition

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Abstract— Interpolation spaces of functions play a major role in the modern mathematics and mathematical physics. Holder-Lipschitz condition is one of the fundamental concepts of this realm. This concept is quite simple but very substantial. The purpose of this paper is to explain a range of ideas behind Holder-Lipschitz condition.

Index Terms— continuity, uniform continuity, module of continuity, Holder-Lipschitz condition, closed set, bounded set, ideal.

1 Introduction

On Holder-Lipschitz condition

Definition:

Let on some set $X \subset R^n$ is given a function $f : X \to \mathrm{R}$, such as:

$$\forall x, y \in X, \exists L > 0$$

and the following inequality is held:

$$|f(x) - f(y)| \le L\rho^{\lambda}(x, y)$$

In case $n = 1$ the condition turns into: $|f(x) - f(y)| \le L|x - y|^{\lambda}$

In case $\lambda = 1$ the condition is called Lipschitz condition, in case $0 < \lambda < 1$ - Holder condition.

Remark: If we let $L < 1$ in Lipschitz condition we get contracting mapping.

2 Result and Discussion

Basic properties.

Initially, we will figure out some properties of functions which satisfy Lipchitz condition ($\lambda = 1$).

Claim 1. Function, which satisfies Lipschitz condition on bounded set, is bounded.

Proof.

We need to show that:

$$\exists \mathrm{C} > 0 : \forall \mathrm{x} \in X \ \ |f(x)| \le C \text{ is met.}$$

Notice that $f(x) \equiv \mathrm{C}$ is bounded by a constant C.

Let $f(x)$ be not identical constant and not bounded. Then $\exists x_0, y_0 \in X : f(x_0) \ne f(y_0)$.

As $f(x)$ is unbounded, then $\forall \mathrm{C} > 0 : \exists \mathrm{x} \in X$ so that: $|f(x)| > C$.

Suppose $d = diamX$, let $C = f(x_0) + L(d+1)$, and due to unboundedness $\exists z_0 \in X : |f(z_0)| > C$.

To sum up, we have:

$$|f(z_0) - f(x_0)| \ge |f(x_0) + L(d+1) - f(x_0)| = L(d+1) > L|z_0 - x_0|,$$

which contradicts with Lipschitz condition. ■

Claim 2. Function, which satisfies Lipchitz condition, is uniformly continuous.

Proof.

Let $\delta = \frac{\varepsilon}{L}$, then:

$$\forall \varepsilon > 0 \exists \delta > 0 : \forall x, y \in X : |x - y| \le \delta \Rightarrow |f(x) - f(y)| \le L|x - y| = L\frac{\varepsilon}{L} = \varepsilon \ ■$$

Claim 3. Continuously differentiable function $f : X \to R$ on closed and bounded set $x \in X \subset R^n$ satisfies on it Lipschitz condition and $L = \sup_{x \in X} |f'(x)|$.

Proof.

As $f'(x)$ is continuous function on closed and bounded set X, Weierstrass theorem can be applied and according to the theorem the function $f'(x)$ achieves on X its minimum and maximum values.

Let $L = \sup_{x \in X} |f'(x)| < \infty$. $f(x)$ satisfies Lagrange's theorem. By applying it for arbitrary interval $(y, z) \subset X$ we conclude $\exists \xi \in (y, z) : f(z) - f(y) = f'(\xi)(z - y)$ and $f'(\xi) \le L$.

Finally:

$f(y)-f(z)\le L|z-y| \Rightarrow \forall z,y\in X : |f(z)-f(y)|\le L|z-y|.$ ■

Example.

It will be shown that the following function

$$f(x)=|x|, x\in[a,b]: -\infty<a<b<\infty$$

satisfies Lipschitz condition.

As inequality $||x|-|y||\le|x-y|$ is met, we can conclude that:

$$|f(x)-f(y)|=||x|-|y||\le|x-y|, \text{ where } L=1.$$

More properties.

We suppose that two functions $f(x), g(x)$ and some constant $c\in R$ are given. These two functions satisfy Lipschitz condition with constants L_1 and L_2 respectively. We will show that following functions also satisfy Lipschitz condition:

1. $cf(x)$.

$|cf(x)-cf(y)|=|c||f(x)-f(y)|\le|c|L_1|x-y|.$

As a Lipschitz constant we take $L=|c|L_1$.

2. $\alpha f(x)+\beta g(x)$.

$|\alpha f(y)+\beta g(y)-\alpha f(x)-\beta g(x)|\le|\alpha\{f(y)-f(x)\}+\beta\{g(y)-g(x)\}|\le$

$|\alpha\{f(y)-f(x)\}|+|\beta\{g(y)-g(x)\}|\le|\alpha|L_1|y-x|+|\beta|L_2|y-x|=(|\alpha|L_1+|\beta|L_2)|y-x|$

We take $L=|\alpha|L_1+|\beta|L_2$ as a Lipschitz constant.

3. $f(g(x))$.

$|f(g(x))-f(g(y))|\le L_1|g(x)-g(y)|\le L_1L_2|x-y|.$

In this case as a constant we choose $L=L_1L_2$.

4. $f(x)g(x)$.

To prove that this function satisfies Lipschitz condition, we will use the fact that on bounded set functions $f(x)$ and $g(x)$ are limited by constants C_1 and C_2.

$|f(y)g(y)-f(x)g(x)|=|f(y)g(y)-f(x)g(y)+f(x)g(y)-f(x)g(x)|=$

$=|g(y)\{f(y)-f(x)\}+f(x)\{g(y)-g(x)\}|\le|g(y)||f(y)-f(x)|+|f(x)||g(y)-g(x)|\le$

$C_1K_1|y-x|+C_2K_2|y-x|=(C_1K_1+C_2K_2)|y-x|$

As a Lipschitz constant we take $L=C_1K_1+C_2K_2$.

Additionally, we will figure out some properties of functions which satisfy Holder condition ($0<\lambda<1$).

Claim 4. Function, which satisfies Holder continuity on bounded set, is limited on it.

Proof. Proof is identical as in Lipschitz case. ■

Claim 5. Function, which satisfies Holder condition, is uniformly continuous.

Proof. Let $\delta=\left(\frac{\varepsilon}{K}\right)^{1/\lambda}$, where K>0, then:

$$\forall\varepsilon>0\exists\delta>0:\forall x,y\in X:|x-y|\le\delta\Rightarrow|f(x)-f(y)|\le K|x-y|^{\lambda}\le K\left(\left(\frac{\varepsilon}{K}\right)^{1/\lambda}\right)^{\lambda}=\varepsilon$$ ■

Claim 6. If function satisfies Holder condition then it satisfies Holder condition for any σ lying in the half-interval $(0;\lambda]$.

Proof.

We suppose that $f(x)$ satisfies Holder continuity with λ and constant K.

We will consider two cases:

1. $|y-x|<1$

   In this case $|f(y)-f(x)|\le K|y-x|^{\lambda}\le K|y-x|^{\sigma}$.

2. $|y-x|\ge1$

   As $f(x)$ satisfies Holder condition, it is bounded $\exists C>0:\forall x\in X$ and $|f(x)|\le C$.

   $|f(y)-f(x)|\le|f(y)|+|f(x)|\le\{|f(y)|+|f(x)|\}|y-x|^{\sigma}\le 2C|y-x|^{\sigma}$.

   Thus, we can conclude that, $\forall x,y\in X:|f(y)-f(x)|\le M|y-x|^{\sigma}$, where $M=\max\{K,2C\}$. ■

Finally, we will study cases where $\lambda \notin (0,1]$:

1. $\lambda > 1$

$$0 \le \frac{|f(y) - f(x)|}{|y - x|} \le K|x - y|^{\lambda-1}$$

Let $y = x + \Delta x$

$$0 \le \frac{|f(x+\Delta x) - f(x)|}{\Delta x} \le K|\Delta x|^{\lambda-1}$$

By tending, $\Delta x \to 0$ we get: $f'(x) = 0 \Rightarrow f(x) \equiv C$

2. $\lambda = 0$

$$|f(x) - f(y)| \le K$$

All bounded functions satisfy this condition, however the property of being continuous is lost.
To exemplify, $f(x) = \operatorname{sgn} x, |\operatorname{sgn} x - \operatorname{sgn} y| \le 2$, but $\operatorname{sgn} x$ is not continuous at the point $x_0 = 0$.

3. $\lambda < 0$

$$|f(x) - f(y)| \le \frac{K}{|x - y|^{|\lambda|}}$$

In this case there is no boundedness.

Algebraic properties.

We consider all functions $f : X \to R,$ where R is a set of real numbers. For any $f, g \in R^X$ and for any $r \in R$ we introduce the following operations $+$ (sum) and $\circ$ (product):

1. $(f + g)(x) := f(x) + g(x)$
2. $(f \circ g)(x) := f(x)g(x)$
3. $r(x) := r, \forall x \in X$
4. $(-f)(x) := -(f(x))$

We will show that the introduced sum operations satisfy the axioms of an abelian group.

1. Associativity:

$$((f + g) + h)(x) = (f + g)(x) + h(x) = f(x) + g(x) + h(x) = f(x) + (g + h)(x) = (f + (g + h))(x)$$

2. 0 is an identity element:

$$f(x) + 0 = 0 + f(x) = f(x)$$

3. For every $f(x)$ an inverse element $(-f)(x)$ exists:

$$f(x) + (-f)(x) = f(x) - (f(x)) = 0$$

$$(-f)(x) + f(x) = -(f(x)) + f(x) = 0$$

4. Commutativity:

$$(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)$$

We will go further and show distributivity, associativity, commutativity and exsistence of an identity element for the product operation:

1. Distributivity:

$$(f(g + h))(x) = f(x)(g + h)(x) = f(x)(g(x) + h(x)) = f(x)g(x) + f(x)h(x) = (fg)(x) + (fh)(x)$$

2. Commutativity:

$$(f \circ g)(x) = f(x)g(x) = g(x)f(x) = (g \circ f)(x)$$

3. Associativity:

$$((fg)h)(x) = (fg)(x)h(x) = f(x)g(x)h(x) = f(x)(gh)(x) = (f(gh))(x)$$

4. 1 is an identity element:

$$f(x) \circ 1 = f(x)$$

$$1 \circ f(x) = f(x)$$

Thus, we obtain a commutative ring.

We will study more properties of this ring and some of its subrings.
We consider all continuous functions $f : X \to R$. This set of functions with the introduced above operations form a subring of the ring because sum and product of continuous functions are also continuous functions. Furthermore, we consider all continuous functions $f : [a,b] \to R.$ If we take an ideal I_n of all continuous

functions f such that $f(x)$ equals $0\ \forall x \geq b-(b-a)/n$, we will obtain an increasing sequence of ideals $I_1, I_2, \ldots$. Therefore, this ring is not Noetherian ring. The set of Lipschitz continuous functions form a subring of a ring of continuous functions, as sum and product of Lipschitz continuous functions are also Lipschitz continuous functions. The proof of it was given above. The subring of Lipschitz continuous functions is also not Noetherian one.

Module of continuity.

Module of continuity is a classical tool applied for describing smoothness of functions. The idea of module of continuity is found in a range of applications in theory of approximation, functional spaces and other spheres of modern analysis.

Definition.

A function $\omega(h), h > 0,$ which is defined as $\omega(h) = \sup_{|y-x|<h} |f(y) - f(x)|$ is called module of continuity of $f(x)$.

We will consider a number of simple facts:

1. Function $\omega(h)$ is non-decreasing $\omega(h_1) \leq \omega(h_2), h_1 \leq h_2$

Proof.

$$\omega(h_1) = \sup_{|y-x|<h_1} |f(y) - f(x)|$$

$$\omega(h_2) = \sup_{|y-x|<h_2} |f(y) - f(x)|$$

As upper bound over extended set can only become bigger, we can conclude that:

$$\sup_{|y-x|<h_1} |f(y) - f(x)| \leq \sup_{|y-x|<h_2} |f(y) - f(x)|$$

To finalize,

$$\omega(h_1) = \sup_{|y-x|<h_1} |f(y) - f(x)| \leq \sup_{|y-x|<h_2} |f(y) - f(x)| = \omega(h_2) \ \blacksquare$$

2. $\omega(h_1 + h_2) \leq \omega(h_1) + \omega(h_2)$ *(half-additivity)*

Proof.

$|y - x| < h_1 + h_2$, between x and y $\exists z : |x - z| < h_1, |z - y| < h_2$

$$\omega(h_1 + h_2) = \sup_{|y-x|<h_1+h_2} |f(y) - f(x)| = \sup_{|y-x|<h_1+h_2} |f(y) - f(z) + f(z) - f(x)| \leq$$

$$\leq \sup_{|x-z|<h_1} |f(y) - f(z)| + \sup_{|z-y|<h_2} |f(z) - f(x)| = \omega(h_1) + \omega(h_2) \ \blacksquare$$

3. $\omega(nh) \leq n\omega(h), n \in \mathrm{N}$

Proof.

Basis of induction $n = 1 : \omega(h) \leq \omega(h)$

Induction hypothesis: $\omega(nh) \leq n\omega(h)$

We will proof it for case $n+1$:

$$\omega((n+1)h) \leq \omega(nh) + \omega(h) \leq n\omega(h) + \omega(h) = (n+1)\omega(h) \ \blacksquare$$

4. $\omega(\lambda h) < (1+\lambda)\omega(h), \lambda > 0$

Proof.

$\exists n \in \mathrm{N} : \lambda \leq n < 1 + \lambda$, then: $\omega(\lambda h) \leq \omega(nh) \leq n\omega(h) < (1+\lambda)\omega(h) \ \blacksquare$

5. Connection between uniform continuity and module of continuity is established by the following claim, which is given without proof:

Claim 7. For uniform continuity of function $f(x)$ it is necessary and sufficient that $\lim_{h \to 0} \omega(h) = 0.$

6. Connection between module of continuity and Holder-Lipschitz condition is established by the following claim:

Claim 8. The following statements are equivalent:

1. $f(x)$ satisfies Lipschitz-Holder condition with λ and constant K
2. $\omega(h) \leq Mh^{\lambda}$

Proof.

$1 \Rightarrow 2$: $|f(y) - f(x)| \leq \omega(|y - x|) \leq M|y - x|^{\lambda}$

$2 \Rightarrow 1$: If $|y-x| \le h$ then: $|f(y)-f(x)| \le M|y-x|^{\lambda} \le Mh^{\lambda}$ ■

3 Conclusion

In this paper there were considered some basic properties of functions, which satisfy Lipschitz condition and more deep properties on Holder condition. Moreover, there were considered some algebraic properties and established the link between module of continuity and Holder-Lipschitz condition.

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